Lecture 6

Andrei Antonenko

February 12, 2003

1 Multiplicative inverse

In this lecture we will continue with the properties of matrix operations.

(M4) Existence of the multiplicative inverse. Matrix B is called the **inverse** for the square matrix A if $BA = AB = I$. The existence of the inverse is a very difficult question, which we will solve later. Now we'll simply give some examples, and then a formula for an inverse of 2×2-matrices.

Example 1.1. *The matrix* $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *doesn't have an inverse, since if* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is an inverse, than* $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *from what* $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *which is never the case.*

Example 1.2. *The inverse for the matrix* $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ *is* $\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$ *since*

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now we're ready to give a formula for an inverse of 2×2-matrix.

Proposition 1.3. *The inverse of* 2×2*-matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *exists if and only if* $ad - bc \neq 0$ *and*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Proof. If $ad - bc \neq 0$ then we can simply check that this matrix is the inverse:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} ad - bc & -ab + ab \\ cd - cd & -cb + da \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We will not give a proof that if a matrix has an inverse, then $ad - bc \neq 0$. This fact can be generalized to the case of larger matrices, and we'll prove it later in more general form. $\square$

The proposition above is useful when you want to get an inverse of a 2×2-matrix. Later we'll provide a method of finding the inverses of larger matrices, but for 2×2-matrices this is the easiest one.

2 Transpose of a matrix

Definition 2.1. *Matrix B is called* **transpose** *of a matrix A (notation: $B = A^\top$) if $b_{ij} = a_{ji}$.*

In other words, we should take rows of a matrix A and write them as columns of matrix B. Then $B = A^\top$. In general form, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

then

$$A^\top = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Let's notice, that if A is an $m \times n$-matrix, then $A^\top$ is an $n \times m$-matrix.

Example 2.2. *Let* $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. *Than* $A^\top = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

Example 2.3. *Let* $A = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$. *Than* $A^\top = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$. *In general,*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}^\top = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \text{ and } \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}^\top = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

Example 2.4. *Let* $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. *Than* $A^\top = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$.

Now we'll give some properties of a transpose.

1. $(A^\top)^\top = A$.
2. For any number λ we have $(\lambda A)^\top = \lambda \cdot A^\top$. So, we can first multiply a matrix by a number, and than take a transpose, or first take a transpose and than multiply the result by a number — we'll get the same result.
3. For any equal-size matrices A and B we have $(A+B)^\top = A^\top + B^\top$.
4. For any matrices A and B such that AB is defined we have $(AB)^\top = B^\top A^\top$.

Properties 1-3 are obvious, and the property 4 requires a proof. But first we'll give appropriate examples.

Example 2.5. *Let* $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. *Then* $AB = \begin{pmatrix} 5 & 5 \\ 11 & 11 \end{pmatrix}$, *and* $(AB)^\top$ *equals to* $\begin{pmatrix} 5 & 11 \\ 5 & 11 \end{pmatrix}$. *Now,* $A^\top = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, *and* $B^\top = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$. *So,* $A^\top B^\top = \begin{pmatrix} 4 & 8 \\ 6 & 12 \end{pmatrix}$, *and* $B^\top A^\top = \begin{pmatrix} 5 & 11 \\ 5 & 11 \end{pmatrix}$. *So we see, that* $(AB)^\top = B^\top A^\top$ *and* $(AB)^\top \neq A^\top B^\top$.

Proof of the property 4. Let $AB = C = (c_{ik})$. Then

$$c^\top_{ki} = c_{ik} = \sum_j a_{ij} b_{jk} = \sum_j b^\top_{kj} a^\top_{ji},$$

and so $C^\top = B^\top A^\top$. □

3 Matrix of the system of linear equations

Let's consider the system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = & b_2 \\ & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = & b_m \end{cases}$$

Then with this system we can associate an $m \times (n+1)$-matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

As for systems, for matrices we can define elementary operations, REF, RREF, and use the same algorithms as for systems to reduce matrices to REF and RREF. So, given a system, we can write its matrix, and then perform all the operations to transpose it to some of these forms, and then get back to the system and write the solution for it.

Let's consider a linear system:

$$\left\{ \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array} \right. \tag{1}$$

We can write 3 following matrices associated with this system:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{2}$$

Now we can see that the system (1) can be written in the following form:

$$AX = B \tag{3}$$

Let's check it:

$$AX = B \Longleftrightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \Longleftrightarrow$$

$$\begin{pmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} \cdot x_n \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \cdots + a_{2n} \cdot x_n \\ \cdots \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \cdots + a_{mn} \cdot x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \Longleftrightarrow$$

$$\left\{ \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array} \right.$$

4 Elementary row operations in matrix form

Now, we'll consider elementary row operations in matrix form.

4.1 Interchanging

Let's imagine that we want to interchange two rows if a matrix. It can be done by multiplying by the appropriate matrix. So, if we want to interchange rows i and j then we should use the following matrix:

$$\begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array}\begin{pmatrix} 1 & & \vdots & & \vdots & \\ & \ddots & \vdots & & \vdots & \\ \cdots\cdots & & 0 & \cdots & 1 & \cdots\cdots \\ & & \vdots & \ddots & \vdots & \\ \cdots\cdots & & 1 & \cdots & 0 & \cdots\cdots \\ & & \vdots & & \vdots & \ddots \\ & & \vdots & & \vdots & & 1 \end{pmatrix} = P_{ij}$$

where all the elements that are not shown are the same as the elements of the identity matrix. If we multiply this matrix by A, i.e. take $P_{ij}A$, we'll get:

$$P_{ij}A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \multicolumn{4}{c}{\dotfill} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \multicolumn{4}{c}{\dotfill} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ for } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \multicolumn{4}{c}{\dotfill} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \multicolumn{4}{c}{\dotfill} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

4.2 Multiplication

Let's imagine that we want to multiply a row of a matrix by a given number $c \neq 0$. It can be done by multiplying by the appropriate matrix. So, if we want to multiply row i by $c \neq 0$ then we should use the following matrix:

$$\begin{array}{c} \\ \\ i \\ \\ \\ \end{array}\begin{pmatrix} 1 & & \vdots & & \\ & \ddots & \vdots & & \\ \cdots\cdots & & c & \cdots\cdots & \\ & & \vdots & \ddots & \\ & & \vdots & & 1 \end{pmatrix} = Q_i(c)$$

where all the elements that are not shown are the same as the elements of the identity matrix. If we multiply this matrix by A, i.e. take $Q_i(c)A$, we'll get:

$$Q_i(c)A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \hdotsfor{4} \\ ca_{i1} & ca_{i2} & \cdots & ca_{in} \\ \hdotsfor{4} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ for } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \hdotsfor{4} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \hdotsfor{4} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

4.3 Addition

Let's imagine that we want to add one row multiplied by a number to some other row. It can be done by multiplying by the appropriate matrix. So, if we want to add row j multiplied by c to row i then we should use the following matrix:

$$\begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} \begin{pmatrix} 1 & & & & \vdots & & \\ & \ddots & & & \vdots & & \\ \cdots\cdots & & 1 & \cdots & c & \cdots\cdots & \\ & & & \ddots & \vdots & & \\ & & & & 1 & & \\ & & & & \vdots & \ddots & \\ & & & & \vdots & & 1 \end{pmatrix} = I + cI_{ij}{}^{1}$$

where all the elements that are not shown are the same as the elements of the identity matrix. If we multiply this matrix by A, i.e. take $(I + cI_{ij})A$, we'll get:

$$(I + cI_{ij})A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \hdotsfor{4} \\ a_{i1} + ca_{j1} & a_{i2} + ca_{j2} & \cdots & a_{in} + ca_{jn} \\ \hdotsfor{4} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \hdotsfor{4} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ for } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \hdotsfor{4} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \hdotsfor{4} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \hdotsfor{4} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Now we'll give some examples.

Example 4.1. *Let's suppose we want to interchange the 2nd and the 3rd rows of the matrix*

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

[1] by I_{ij} we will denote the matrix which has 1 at (i, j)-th place and zeros on all other places.

We will do it using matrix P_{23}:

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}=\begin{pmatrix}1&2&3\\7&8&9\\4&5&6\end{pmatrix}$$

Example 4.2. *Let's suppose we want to multiply the 2nd row of the matrix*

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$$

by 4. We will do it using matrix $Q_2(4)$:

$$\begin{pmatrix}1&0&0\\0&4&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}=\begin{pmatrix}1&2&3\\4\cdot 4&4\cdot 5&4\cdot 6\\7&8&9\end{pmatrix}=\begin{pmatrix}1&2&3\\16&20&24\\7&8&9\end{pmatrix}$$

Example 4.3. *Let's suppose we want to add the 1st row of the matrix*

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$$

multiplied by 3 to the 3rd row. We will do it using matrix $I+I_{31}$:

$$\begin{pmatrix}1&0&0\\0&1&0\\3&0&1\end{pmatrix}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}=\begin{pmatrix}1&2&3\\4&5&6\\3\cdot 1+7&3\cdot 2+8&3\cdot 3+9\end{pmatrix}=\begin{pmatrix}1&2&3\\4&5&6\\10&14&18\end{pmatrix}$$

So, as we can see, all elementary row operations can be considered as a multiplication by the appropriate matrix.